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The suppression of short waves by a train of long waves

By A. M. BALK
Applied Mathematics 217-50, California Institute of Technology, Pasadena, CA 91125, USA

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It is shown that a train of long waves can suppress a short-wave field due to four-wave resonance interactions. These interactions lead to the diffusion (in Fourier space) of the wave action of the short-wave field, so that the wave action is transported to the regions of higher wavenumbers, where it dissipates more effectively. The diffusion equation is derived.

1. Introduction

It has been noticed in several experiments during the last 30 years (Mitsuyasu 1966; Banner 1973; Phillips & Banner 1974; Yuen 1988) that a long wave can suppress short waves: the energy density of a short-wave field can decrease when a train of longer waves propagates into the short-wave field: 'the short waves can be swept clean by the long wave'.

This surprising phenomenon was observed in laboratory experiments with wind-generated water waves. Although its physical mechanism was not entirely understood, it was connected to the drift current (under the water surface) induced by the wind.

The aim of this paper is to show that a train of long waves can suppress short waves merely due to the four-wave resonance interactions of small-amplitude waves, so that the presence of the drift current is not required, and the phenomenon can occur in continuous media of various physical natures. To be more specific, we will consider two-dimensional media with the dispersion laws

$$\omega(\boldsymbol{k}) = \text{const}\,|\boldsymbol{k}|^{\alpha}, \quad 0 < \alpha < 1, \qquad \dim \boldsymbol{k} = 2, \tag{1.1}$$

although the reasoning of this paper can be applied to other media. A distinct representative of the class (1.1) is the system of deep-water gravity waves with the dispersion law

$$\omega(\boldsymbol{k}) = (g|\boldsymbol{k}|)^{1/2} \tag{1.2}$$

(g is the gravity acceleration).

The dispersion laws (1.1) give that long waves travel faster than short waves, so that a long wave can, indeed, propagate through a short-wave field. Also, the dispersion laws (1.1) do not allow three-wave resonance interactions, so that the resonances of the lowest-order involve four waves. In general, there are various four-wave resonances:

$$\pm\boldsymbol{k}_1 \pm \boldsymbol{k}_2 \pm \boldsymbol{k}_3 \pm \boldsymbol{k}_4 = 0, \qquad \pm\omega(\boldsymbol{k}_1) \pm \omega(\boldsymbol{k}_2) \pm \omega(\boldsymbol{k}_3) \pm \omega(\boldsymbol{k}_4) = 0, \tag{1.3}$$

which are defined by different choices of signs in (1.3). If the dispersion law $\omega(\boldsymbol{k})$ is a positive function, the system (1.3) when all the signs are the same has no solutions. The dispersion law (1.1) also does not allow four-wave interactions (1.3) with three signs the same, i.e. decay of a wave into three other waves (these are forbidden due to the same reasons, the 3-wave processes of decay of a wave into two other waves are forbidden). The system of resonance equations (1.3) has solutions only in the case of two + signs and two − signs, when it can be written in the form

$$\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}_3+\boldsymbol{k}_4, \qquad \omega(\boldsymbol{k}_1)+\omega(\boldsymbol{k}_2)=\omega(\boldsymbol{k}_3)+\omega(\boldsymbol{k}_4). \tag{1.4}$$

It is well known that four-wave resonance interactions lead to the *Benjamin-Feir instability* (Benjamin & Feir 1967), or the *decay instability of the second kind* (Zakharov 1968), when a wave train (with a basic wave vector $\boldsymbol{p}$) disintegrates into other waves by means of four-wave resonance interactions:

$$2\boldsymbol{p}=\boldsymbol{k}_1+\boldsymbol{k}_2, \qquad 2\omega(\boldsymbol{p})=\omega(\boldsymbol{k}_1)+\omega(\boldsymbol{k}_2). \tag{1.5}$$

If the dispersion law $\omega(\boldsymbol{k})$ is a non-negative function, then, obviously, the frequencies of the waves that arise $\omega_i=\omega(\boldsymbol{k}_i)$ $(i=1,2)$ cannot exceed double the frequency of the original wave. In fact, the limitation is more strict. For example, in the case of gravity waves (1.2), it follows from (1.5) that $\omega_i \leqslant \frac{3}{2}\omega$, and therefore, $|\boldsymbol{k}_i| \leqslant \frac{9}{4}|\boldsymbol{p}|$ $(i=1,2)$. Thus, the interactions (1.5) do not affect short waves (with wavenumbers $|\boldsymbol{k}|$ larger than some given value).

Does the train of long waves affect the short-wave field by means of four-wave resonance interactions?

In (1.5) the wave vector $\boldsymbol{p}$ of the long wave enters one side of the equations. However, there are four-wave interactions (1.4) of another kind, when the wave vectors of the long wave $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ enter both sides of the resonance equations:

$$\boldsymbol{p}_1+\boldsymbol{k}_1=\boldsymbol{p}_2+\boldsymbol{k}_2, \qquad \omega(\boldsymbol{p}_1)+\omega(\boldsymbol{k}_1)=\omega(\boldsymbol{p}_2)+\omega(\boldsymbol{k}_2). \tag{1.6}$$

Here $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are wave vectors of short waves. These interactions affect short waves with arbitrarily large wavenumbers k.

If the train of long waves were monochromatic, then $\boldsymbol{p}_1=\boldsymbol{p}_2$, and by virtue of (1.6), $\boldsymbol{k}_1=\boldsymbol{k}_2$, so that the interactions (1.6) could not change the energy spectrum of the short-wave field. However, the train of long waves is not monochromatic and consists of waves with wave vectors $\boldsymbol{p}$ from a neighbourhood of some fixed vector $\boldsymbol{p}_0$ – the basic wave vector of the wave train (otherwise the long wave would be infinite in space and would be present in the region of short waves during an infinite time). Thus, the wave vectors $\boldsymbol{k}_1, \boldsymbol{k}_2$ are not the same, but differ from each other by a 'small' vector $\boldsymbol{k}_2-\boldsymbol{k}_1=\boldsymbol{q}=\boldsymbol{p}_1-\boldsymbol{p}_2$, where $|\boldsymbol{q}| \ll |\boldsymbol{p}_0|$.

We will see that the interactions (1.6) lead to diffusion in $\boldsymbol{k}$-space of the wave action $n_{\boldsymbol{k}}$ of the short-wave field†. This diffusion is one-dimensional and occurs along the curves

$$\Omega(\boldsymbol{k})=C=\text{const} \tag{1.7}$$

where

$$\Omega(\boldsymbol{k})=\omega(\boldsymbol{k})-\frac{\partial\omega}{\partial\boldsymbol{p}}(\boldsymbol{p}_0)\cdot\boldsymbol{k} \tag{1.8}$$

is the dispersion law of the short waves in the frame of reference moving with the

† The wave action distribution $n_{\boldsymbol{k}}$, or the wave action spectrum, is related to the energy spectrum $\varepsilon_{\boldsymbol{k}}$ by the formula $n_{\boldsymbol{k}}=\varepsilon_{\boldsymbol{k}}/\omega(\boldsymbol{k})$.

group velocity of the train of long waves. The evolution of the function n_k on each of the curves (1.7) occurs independently of the values of this function on other curves. The redistribution of the wave action between the curves (1.7) is a slower process and is neglected.

The diffusion leads to the equipartion of the wave action on each of the curves (1.7). In the absence of dissipation, the total wave action on each of the curves (1.7)

$$\mathcal{N}(C) = \int n_k \delta(\Omega(\boldsymbol{k}) - C) \mathrm{d}\boldsymbol{k} \tag{1.9}$$

remains constant, and the distribution n_k on each curve approaches the mean value

$$\bar{n}(C) = \frac{\mathcal{N}(C)}{\int \delta(\Omega(\boldsymbol{k}) - C) \mathrm{d}\boldsymbol{k}} .$$

It is crucial whether the curves (1.7) are closed or go to infinity. If the curves are closed, then the mean value $\bar{n}(C)$ is non-zero; it is determined by the initial conditions. However, if the curves are not closed and have an infinite length, the distribution n_k vanishes with time. Then one can say that the long wave is sweeping the short waves to the region of large $|\boldsymbol{k}|$ where they dissipate, and thereby, the long wave suppresses the short-wave field.

Consider, for example, the gravity waves (1.2). Without loss of generality, we can assume that $g = 1$, $\boldsymbol{p}_0 = (1, 0)$. Then

$$\Omega(\boldsymbol{k}) = (\xi^2 + \eta^2)^{1/4} - \tfrac{1}{2}\xi$$

where ξ, η are components of the wave vector $\boldsymbol{k}$ (respectively along and across the direction of propagation of the long wave). The corresponding curves (1.7) are shown in figure 1. There are some closed curves (1.7), but they are located inside the circle $|\boldsymbol{k}| \leqslant |\boldsymbol{p}_0|$. If $|\boldsymbol{k}| > |\boldsymbol{p}_0|$, then the wave vector $\boldsymbol{k}$ belongs to one of the curves that are not closed and go to infinity. The wave action will vanish with time on these curves.

It is interesting to note that for dispersion laws of the form (1.1) with $\alpha > 1$, the curves (1.7) are closed, and the diffusion makes the spectrum equal to a non-zero constant on each of these curves. Actually, this happens for any dispersion law $\omega(k)$ which depends only on the modulus of the wave vector and grows faster than k as $k \to \infty$; then for sufficiently large k, $\Omega(\boldsymbol{k}) = \omega(k)$ and the curves (1.7) are circles k =const. In this situation, the diffusion leads to the isotropization of the short-wave spectrum n_k with sufficiently large k, but not to its suppression. In particular, this is true for the gravity–capillary waves with the dispersion law

$$\omega(\boldsymbol{k}) = \left(gk + \frac{\sigma}{\rho} k^3 \right)^{1/2} \tag{1.10}$$

(σ is the surface tension coefficient, ρ is the fluid's density). However, normally for water the capillarity effects are important only for waves of a few centimetres and shorter, while in the above-mentioned experiments the wavelength is of the order of tens of centimetres or metres. Thus, the diffusion will lead to the transport of the wave action to the capillarity region (where the dissipation is higher) and then to isotropization. Also, when capillarity becomes important, three-wave interactions become possible and dominate the nonlinear wave dynamics. In this region our reasoning is not applicable.

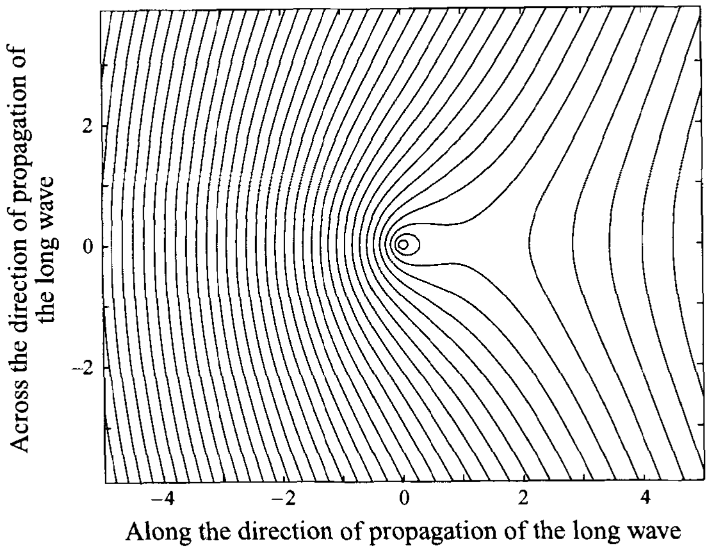

FIGURE 1. The contour lines of the function $\Omega(\boldsymbol{k}) = (\xi^2 + \eta^2)^{1/4} - \frac{1}{2}\xi$. In the case of deep-water gravity waves the diffusion of $n_{\boldsymbol{k}}$ occurs in the $\boldsymbol{k} = (\xi, \eta)$-plane along these lines.

2. Diffusion in the k-plane

2.1. *Wave kinetic equation*

If the dispersion law of the medium forbids three-wave resonance interactions (like the dispersion laws (1.1)), and the wave amplitudes are small enough, then in the leading order the evolution of the wave action distribution $n_{\boldsymbol{k}}$ is described by the four-wave kinetic equation (Hasselmann 1962; Zakharov & Filonenko 1966; Crawford, Saffman & Yuen 1980)

$$\frac{\partial n_1}{\partial t} = \int R_{1234}\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \times [n_1 n_3 n_4 + n_2 n_3 n_4 - n_1 n_2 n_3 - n_1 n_2 n_4]\mathrm{d}\boldsymbol{k}_2\mathrm{d}\boldsymbol{k}_3\mathrm{d}\boldsymbol{k}_4, \quad (2.1)$$

where $\omega_i = \omega(\boldsymbol{k}_i)$, $n_i = n_{\boldsymbol{k}_i}$ $(i = 1, 2, 3, 4)$; $R_{1234} = R(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4)$ is a positive function, which is determined by the coefficients of the medium's equation in the Fourier representation. If this equation is written in the Hamiltonian form, the kernel R is defined in the following way.

When three-wave resonance interactions are not allowed, quadratic terms in the equation of the medium can be excluded with the aid of a certain canonical transformation. Some of the cubic terms, corresponding to the resonances (1.3) which are not allowed by the dispersion law, can also be excluded with the aid of a canonical transformation, so that in the new variables the medium is described by a cubic Hamiltonian equation with the following fourth-degree Hamiltonian (Zakharov, Musher & Rubenchik 1985):

$$H = \int \omega(\boldsymbol{k}) a_{\boldsymbol{k}} a^*_{\boldsymbol{k}} d\boldsymbol{k} + \frac{1}{4}\int W_{1234}\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4) a_{\boldsymbol{k}_1} a_{\boldsymbol{k}_2} a^*_{\boldsymbol{k}_3} a^*_{\boldsymbol{k}_4} \mathrm{d}\boldsymbol{k}_1\mathrm{d}\boldsymbol{k}_2\mathrm{d}\boldsymbol{k}_3\mathrm{d}\boldsymbol{k}_4 \quad (2.2)$$

corresponding to the interactions (1.4). The function $W_{1234} = W(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4)$ – the so-called *matrix element* – defines the kernel in the kinetic equation:

$$R_{1234} = \pi \, |W_{1234}|^2 . \tag{2.3}$$

The wave action spectrum n_k is the result of statistical averaging of the canonical wave amplitudes: $\langle a_k^* a_{k'} \rangle = n_k \delta(\boldsymbol{k} - \boldsymbol{k}')$.

Now let us assume that the wave action distribution n_k has a peak at $\boldsymbol{k} = \boldsymbol{p}_0$ (the amplitudes of the short waves are much smaller than the amplitude of the long wave, characterized by the wave vector $\boldsymbol{p}_0$), so that the main contribution to the integral (2.1) comes from the region where the vector $\boldsymbol{k}_2$ and one of the vectors $\boldsymbol{k}_3$ or $\boldsymbol{k}_4$ are 'close' to the vector $\boldsymbol{p}_0$. Since the integrand in (2.1) is symmetric with respect to the exchange $\boldsymbol{k}_3 \leftrightarrow \boldsymbol{k}_4$ we can assume in (2.1) that $\boldsymbol{k}_4$ is close to $\boldsymbol{p}_0$, while $\boldsymbol{k}_3$ is close to $\boldsymbol{k}_1$, and take into account the other possibility ($\boldsymbol{k}_3 \approx \boldsymbol{p}_0$, $\boldsymbol{k}_4 \approx \boldsymbol{k}_1$) by a factor of 2 in front of the integral. Hereafter let us denote by $\boldsymbol{p}$ (with subindices) the wave vectors of long waves, and by $\boldsymbol{k}$, the wave vectors of short waves. In accordance with this, let us rename the variables of integration in (2.1) from $\boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4$ to $\boldsymbol{p}_1, \boldsymbol{k}_2, \boldsymbol{p}_2$ respectively. Then the kinetic equation (2.1) takes the form

$$\begin{aligned}\frac{\partial n_{k_1}}{\partial t} = 2 \int & R(\boldsymbol{k}_1, \boldsymbol{p}_1, \boldsymbol{k}_2, \boldsymbol{p}_2) \, \delta(\boldsymbol{k}_1 + \boldsymbol{p}_1 - \boldsymbol{k}_2 - \boldsymbol{p}_2) \\ & \times \delta(\omega(\boldsymbol{k}_1) + \omega(\boldsymbol{p}_1) - \omega(\boldsymbol{k}_2) - \omega(\boldsymbol{p}_2)) \\ & \times [n_{k_1} n_{k_2} n_{p_2} + n_{p_1} n_{k_2} n_{p_2} - n_{k_1} n_{p_1} n_{k_2} - n_{k_1} n_{p_1} n_{p_2}] \, \mathrm{d}\boldsymbol{p}_1 \, \mathrm{d}\boldsymbol{p}_2 \, \mathrm{d}\boldsymbol{k}_2 \end{aligned} \tag{2.4}$$

where the integration over $\boldsymbol{p}_1, \boldsymbol{p}_2$ is carried out in some neighborhood of the point $\boldsymbol{p}_1 = \boldsymbol{p}_2 = \boldsymbol{p}_0$.

In equation (2.4) we neglect $n_{k_1} n_{k_2} n_{p_2}$ and $n_{k_1} n_{p_1} n_{k_2}$ in comparison with $n_{p_1} n_{k_2} n_{p_2}$ and $n_{k_1} n_{p_1} n_{p_2}$, since n_{k_1} and n_{k_2} are much less than n_{p_1} and n_{p_2}.

Since $|\boldsymbol{p}_1 - \boldsymbol{p}_2| \ll |\boldsymbol{p}_0|$ in (2.4), we can expand

$$\omega(\boldsymbol{p}_1) - \omega(\boldsymbol{p}_2) = \frac{\partial \omega}{\partial \boldsymbol{p}}(\boldsymbol{p}_0) \cdot (\boldsymbol{p}_1 - \boldsymbol{p}_2) = -\frac{\partial \omega}{\partial \boldsymbol{p}}(\boldsymbol{p}_0) \cdot (\boldsymbol{k}_1 - \boldsymbol{k}_2),$$

so that

$$\omega(\boldsymbol{k}_1) + \omega(\boldsymbol{p}_1) - \omega(\boldsymbol{k}_2) - \omega(\boldsymbol{p}_2) = \Omega(\boldsymbol{k}_1) - \Omega(\boldsymbol{k}_2)$$

(for the definition of the function $\Omega(\boldsymbol{k})$, see (1.8)). Thus, (2.4) acquires the form

$$\begin{aligned}\frac{\partial n_{k_1}}{\partial t} = 2 \int & R(\boldsymbol{k}_1, \boldsymbol{p}_1, \boldsymbol{k}_2, \boldsymbol{p}_2) \, \delta(\boldsymbol{k}_1 + \boldsymbol{p}_1 - \boldsymbol{k}_2 - \boldsymbol{p}_2) \\ & \times \delta(\Omega(\boldsymbol{k}_1) - \Omega(\boldsymbol{k}_2)) \, n_{p_1} n_{p_2} (n_{k_2} - n_{k_1}) \, \mathrm{d}\boldsymbol{p}_1 \, \mathrm{d}\boldsymbol{k}_2 \, \mathrm{d}\boldsymbol{k}_2 . \end{aligned} \tag{2.5}$$

This equation shows that the wave action spectrum n_k of the short-wave field evolves along the curves (1.7): owing to the presence of the δ-function $\delta(\Omega(\boldsymbol{k}_1) - \Omega(\boldsymbol{k}_2))$ in (2.5), the time derivative of the spectrum n_k on each curve (1.7) depends only on the values of the spectrum n_k on the same curve; it is independent of the spectrum n_k on other curves (1.7). Such an evolution along certain curves in $\boldsymbol{k}$-space resembles the non-local turbulence of Rossby waves (Balk, Nazarenko & Zakharov 1990).

2.2. *Diffusion equation*

Now, we will reduce (2.5) to the diffusion equation (2.11), which describes the diffusion of the wave action n_k along the curves (1.7) in the $\boldsymbol{k}$-plane.

To simplify the derivation, let us multiply equation (2.5) by a test function $\chi(\boldsymbol{k}_1)$ and integrate over $\mathrm{d}\boldsymbol{k}_1$. After symmetrization in the integral on the right-hand side (which takes away the factor 2) we will arrive at the equation

$$\int \frac{\partial n_{k_1}}{\partial t}\chi(\boldsymbol{k}_1)\mathrm{d}\boldsymbol{k}_1 = \int R(\boldsymbol{k}_1,\boldsymbol{p}_1,\boldsymbol{k}_2,\boldsymbol{p}_2)\delta(\boldsymbol{k}_1+\boldsymbol{p}_1-\boldsymbol{k}_2-\boldsymbol{p}_2)$$
$$\times\delta(\Omega(\boldsymbol{k}_2)-\Omega(\boldsymbol{k}_1))\, n_{p_1}n_{p_2}\,[n_{k_2}-n_{k_1}]\,[\chi(\boldsymbol{k}_1)-\chi(\boldsymbol{k}_2)]\,\mathrm{d}\boldsymbol{p}_1\,\mathrm{d}\boldsymbol{p}_2\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2. \quad (2.6)$$

We denote the components of the wave vectors $\boldsymbol{k}$ by ξ and η (respectively along and across the vector $\boldsymbol{p}_0$; $\boldsymbol{k}_i = (\xi_i,\eta_i)$, $i = 1,2$) and change the variables of integration from ξ_i to $\Omega_i = \Omega(\boldsymbol{k}_i)$, so that

$$\boldsymbol{k}_i = (\xi(\Omega_i,\eta_i),\eta_i), \quad \mathrm{d}\boldsymbol{k}_i = \mathrm{d}\xi_i\mathrm{d}\eta_i = |\partial\Omega_i/\partial\xi_i|^{-1}\mathrm{d}\Omega_i\mathrm{d}\eta_i \quad (i=1,2),$$

and the wave action spectrum $n_{\boldsymbol{k}}(t)$ can be considered as a function of Ω,η,t. Then the integration with respect to Ω_2 can be readily done at the expense of the δ-function $\delta(\Omega_2-\Omega_1)$, and we can use the following expansions in (2.6):

$$n_{k_2}-n_{k_1} = (\eta_2-\eta_1)\left[\left(\frac{\partial n}{\partial \eta}\right)_\Omega\right]_{\Omega=\Omega_1,\eta=\eta_1},$$

$$\chi(\boldsymbol{k}_2)-\chi(\boldsymbol{k}_1) = (\eta_2-\eta_1)\left[\left(\frac{\partial \chi}{\partial \eta}\right)_\Omega\right]_{\Omega=\Omega_1,\eta=\eta_1}$$

(the subscript Ω shows that the derivatives with respect to η are calculated while Ω is held fixed; the derivative with respect to Ω does not enter these expansions since $\Omega_1-\Omega_2=0$).

If the kernel $R(\boldsymbol{k}_1,\boldsymbol{p}_1,\boldsymbol{k}_2,\boldsymbol{p}_2)$ is a continuous function, it can be replaced in (2.6) by $R(\boldsymbol{k}_1,\boldsymbol{p}_0,\boldsymbol{k}_1,\boldsymbol{p}_0)$.

We make the following change of variables of integration:

$$\boldsymbol{p}_1 = \boldsymbol{p}+\boldsymbol{q}/2,\ \boldsymbol{p}_2 = \boldsymbol{p}-\boldsymbol{q}/2,\ \mathrm{d}\boldsymbol{p}_1\mathrm{d}\boldsymbol{p}_2 = \mathrm{d}\boldsymbol{p}\mathrm{d}\boldsymbol{q}$$

and write Ω,η, and $\boldsymbol{k}$ instead of Ω_1,η_1, and $\boldsymbol{k}_1$ respectively:

$$\int \frac{\partial n(\Omega,\eta,t)}{\partial t}\chi(\Omega,\eta)\left|\frac{\partial\Omega}{\partial\xi}\right|^{-1}\mathrm{d}\Omega\mathrm{d}\eta = -\int R(\boldsymbol{k},\boldsymbol{p}_0,\boldsymbol{k},\boldsymbol{p}_0)\left|\frac{\partial\Omega}{\partial\xi}\right|^{-2}$$
$$\times\delta(\xi(\Omega,\eta)-\xi(\Omega,\eta_2)+q_x)\,\delta(\eta-\eta_2+q_y)$$
$$\times n_{\boldsymbol{p}+\boldsymbol{q}/2}\,n_{\boldsymbol{p}-\boldsymbol{q}/2}\,(\eta_2-\eta)^2\left(\frac{\partial n}{\partial\eta}\right)_\Omega\left(\frac{\partial\chi}{\partial\eta}\right)_\Omega \mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\eta_2 \quad (2.7)$$

(q_x, q_y are components of the vector $\boldsymbol{q}$). Finally, we make the change of variables $u=\eta_2-\eta$, $\mathrm{d}u = \mathrm{d}\eta_2$ and integrate over $\boldsymbol{q}$ at the expense of the δ-functions in (2.7). Then

$$\int \frac{\partial n(\Omega,\eta,t)}{\partial t}\chi(\Omega,\eta)\left|\frac{\partial\Omega}{\partial\xi}\right|^{-1}\mathrm{d}\Omega\mathrm{d}\eta = -\int D(\Omega,\eta)\left(\frac{\partial n}{\partial\eta}\right)_\Omega\left(\frac{\partial\chi}{\partial\eta}\right)_\Omega \quad (2.8)$$

where

$$D(\Omega,\eta) = R(\boldsymbol{k},\boldsymbol{p}_0,\boldsymbol{k},\boldsymbol{p}_0)\left|\frac{\partial\Omega}{\partial\xi}\right|^{-2}\int u^2 n_{\boldsymbol{p}+\boldsymbol{q}/2}n_{\boldsymbol{p}-\boldsymbol{q}/2}\mathrm{d}\boldsymbol{p}\mathrm{d}u, \quad (2.9)$$

$$\boldsymbol{q} = \left(u\left(\frac{\partial\xi}{\partial\eta}\right)_\Omega,\ u\right) = \left(-u\frac{\partial\Omega/\partial\eta}{\partial\Omega/\partial\xi},\ u\right). \quad (2.10)$$

Integrating by parts the right-hand side of (2.8) and taking into account that $\chi(\Omega,\eta)$

is an arbitrary function, we obtain the diffusion equation

$$\frac{\partial n(\Omega,\eta,t)}{\partial t} = \left|\frac{\partial \Omega}{\partial \xi}\right| \frac{\partial}{\partial \eta}\left[D(\Omega,\eta)\frac{\partial n}{\partial \eta}\right] \tag{2.11}$$

where the derivatives with respect to η are calculated for fixed Ω, and the diffusion coefficient $D(\Omega,\eta)$ is defined by the formulae (2.9) and (2.10).

According to (2.3)

$$R(\boldsymbol{k},\boldsymbol{p}_0,\boldsymbol{k},\boldsymbol{p}_0) = \pi|W(\boldsymbol{k},\boldsymbol{p}_0,\boldsymbol{k},\boldsymbol{p}_0)|^2; \tag{2.12}$$

it is a function of Ω and η, since the vector $\boldsymbol{p}_0$ is held fixed. The function $W(\boldsymbol{k},\boldsymbol{p}_0,\boldsymbol{k},\boldsymbol{p}_0)$ also arises in the formulae for the decay instability (described by the relations (1.5)) (Zakharov *et al.* 1985) and has a simple physical meaning: it defines the frequency shift due to nonlinear interaction with the wave train which has the basic wave vector $\boldsymbol{p}_0$. The total frequency of the wave with wave vector $\boldsymbol{k}$ is

$$\tilde{\omega}_k = \omega_k + W(\boldsymbol{k},\boldsymbol{p}_0,\boldsymbol{k},\boldsymbol{p}_0)N_0 \tag{2.13}$$

where

$$N_0 = \int n_p \mathrm{d}\boldsymbol{p} \tag{2.14}$$

is the total wave action (per unit volume of the physical space) of the train of long waves.

2.3. *A stationary turbulence spectrum n_k defined by the flux of wave action along the curves (1.7)*

If the train of long waves affects the short-wave field for a 'long' time, then a stationary distribution n_k of the short waves can occur. In the presence of some pumping (e.g. due to the wind in the case of gravity waves), this stationary distribution is not the equilibrium spectrum $n_k = 0$. One can check that the diffusion equation (2.11) conserves the wave action (1.9), and the diffusion equation (2.11) has a stationary solution with constant flux of wave action along each of the curves (1.7). If the regions of pumping and dissipation are far from each other in $\boldsymbol{k}$-space, then in the inertial range the stationary spectrum is defined, according to (2.11), by the following first-order linear ordinary differential equation:

$$D(\Omega,\eta)\frac{\partial n}{\partial \eta} = Q(\Omega).$$

Here n is considered to be a function of Ω,η, so that the derivative with respect to η is calculated while Ω is held fixed, and Ω in this equation is a parameter; $Q(\Omega)$ is the flux of the wave action along the curve (1.7) characterized by this value of Ω. Like the Kolmogorov spectrum of hydrodynamic turbulence (Kolmogorov 1941), this spectrum is defined by the fluxes of conserved quantities, although the latter does not have a power-law form. While the Kolmogorov spectrum is defined by a single parameter (the energy flux), the non-equilibrium spectrum n_k is determined by the function $Q(\Omega)$ (the wave action flux along the curves (1.7)), which essentially depends on the form of the pumping.

3. Estimates

Equation (2.11) describes one-dimensional diffusion of the wave action spectrum n_k along the curves (1.7) in the $\boldsymbol{k}$-plane. The diffusion coefficient (2.9) is proportional to the fourth degree of the amplitude of the long wave, so that when the long wave comes into the short-wave field the diffusion is increased and the wave action n_k is transported faster along the curves (1.7) to the region of higher wavenumbers, where the dissipation is larger. In this way the long wave suppresses the short waves.

3.1. *The diffusion time* τ_0

Let us estimate the characteristic time of this diffusion, i.e. the time τ_0 during which the wave distribution n_k spreads along the curves (1.7) distances of the order K, the characteristic wavenumber of the short waves. According to the diffusion equation (2.11),

$$\frac{1}{\tau_0} \simeq \left|\frac{\partial \Omega}{\partial \xi}\right| D \frac{1}{K^2}. \tag{3.1}$$

We write $\simeq$ instead of $=$ when the left-hand side differs from the right-hand side by a numerical dimensionless constant.

For simplicity let us assume that the waves from the train are much longer than the short waves ($p_0 \ll K$). For the dispersion laws (1.1), the group velocity of long waves is larger than the group velocity of short waves, and under the condition $p_0 \ll K$, we have

$$\left(\frac{\partial \Omega}{\partial \xi}, \frac{\partial \Omega}{\partial \eta}\right) \equiv \frac{\partial \Omega}{\partial \boldsymbol{k}} \equiv \frac{\partial \omega}{\partial \boldsymbol{k}}(\boldsymbol{k}) - \frac{\partial \omega}{\partial \boldsymbol{p}}(\boldsymbol{p}_0) = (-\mu_0, 0) \tag{3.2}$$

where

$$\mu_0 = \left|\frac{\partial \omega}{\partial \boldsymbol{p}}(\boldsymbol{p}_0)\right|$$

is the absolute value of the group velocity of the train of long waves.

This means that the curves (1.7) are the straight lines $\eta = \text{const}$ (cf. figure 1), and the derivatives with respect to η calculated while either Ω or ξ is held fixed are the same. Besides, the first component of the vector (2.10) equals zero: $\boldsymbol{q} = (0, u)$.

From (3.1), (3.2) we have

$$\tau_0 \simeq \frac{K^2}{\mu_0 D}.$$

Assuming that the train of long waves is characterized by the basic wave vector $\boldsymbol{p}_0$, by the spectral width A along $\boldsymbol{p}_0$ and B across $\boldsymbol{p}_0$, and by the magnitude n_0, so that the total wave action (2.14) of the train of long waves is $N_0 \simeq n_0 AB$, we find an estimate for the diffusion coefficient (2.9):

$$D = |W_0|^2 \mu_0^{-2} \int u^2 n(p_x, p_y + \tfrac{1}{2}u) n(p_x, p_y - \tfrac{1}{2}u) \mathrm{d}p_x \mathrm{d}p_y \mathrm{d}u$$
$$\simeq |W_0|^2 \mu_0^{-2} n_0^2 A B^4 \simeq |W_0|^2 \mu_0^{-2} N_0^2 \frac{B^2}{A},$$

where $|W_0|$ is the characteristic magnitude of the matrix element $W(\boldsymbol{k}, \boldsymbol{p}_0, \boldsymbol{k}, \boldsymbol{p}_0)$ (when two of its wave vectors correspond to long waves from the train and the other two to waves from the short-wave field). Thus, the diffusion time is estimated as

$$\tau_0 \simeq \frac{A}{\mu_0 B^2} \theta_0^{-2} \tag{3.3}$$

where

$$\theta_0 \simeq \frac{|W_0|N_0}{\mu_0 K} \tag{3.4}$$

is a dimensionless quantity and might be considered as a parameter of nonlinearity (see (2.13)).

It is interesting that in the case of deep-water gravity waves, this parameter θ_0, and consequently the diffusion time τ_0, does not depend on the characteristic wavenumber K of the short-wave field, since for gravity waves

$$|W_0| \simeq |W(\boldsymbol{k},\boldsymbol{p}_0,\boldsymbol{k},\boldsymbol{p}_0)| \simeq K p_0^2 \tag{3.5}$$

(Zakharov 1992, formula (4.18)), and the wavenumber K cancels out in (3.4).

The formula (3.5) can be understood from considerations of dimension. Indeed, in the case of deep-water gravity waves, it follows from the dimensional considerations that the matrix element W is a homogeneous function of third degree in the wavenumbers. Since the parameter of nonlinearity is an amplitude of a wave times its wave number, then in accordance with (2.13), the matrix element $W(\boldsymbol{k},\boldsymbol{p}_0,\boldsymbol{k},\boldsymbol{p}_0)$ is proportional to p_0^2. To have the third degree of homogeneity, the function $W(\boldsymbol{k},\boldsymbol{p}_0,\boldsymbol{k},\boldsymbol{p}_0)$ should also be proportional to k. Thus, we have (3.5).

For gravity waves $\mu_0 = \frac{1}{2}(g/p_0)^{1/2}$, and the formula (3.4) takes the form

$$\theta_0 \simeq p_0^{5/2} N_0 g^{-1/2} = p_0^2 \frac{\omega_0 N_0}{g}. \tag{3.6}$$

Here $\omega_0 = (gp_0)^{1/2}$ is the frequency of the long wave, and $\omega_0 N_0$ is its energy; it is proportional to gh_0^2, where h_0 is the height of the long wave. Therefore, parameter θ_0 in the case of gravity waves is the squared slope of the long wave: $\theta_0 = (p_0 h_0)^2$.

3.2. *The characteristic time of the 'close' scale interactions τ*

The derivation of the diffusion equation in §2 is based on the assumption that the main contribution to the integral (2.1) comes from the region where the vector $\boldsymbol{k}_2$ and one of the vectors $\boldsymbol{k}_3$ or $\boldsymbol{k}_4$ are 'close' to the vector $\boldsymbol{p}_0$ (corresponding to interactions (1.6) with two long waves) while other interactions can be neglected, including close scale interactions, which are the most important in the Kolmogorov-type turbulence. This assumption is valid if the amplitude of the long wave is 'large enough'. How large should it be? This essentially depends on the form of the spectrum $n_{\boldsymbol{k}}$. Let us, however, make a crude estimate of the characteristic time τ of the spectrum evolution due to the close scale interactions. According to the kinetic equation (2.1),

$$\frac{1}{\tau} \simeq \int |W|^2 \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) n_3 n_4 \mathrm{d}\boldsymbol{k}_3 \mathrm{d}\boldsymbol{k}_4 \simeq \frac{|W|^2 N^2}{\omega} \tag{3.7}$$

where $|W| \simeq K^3$ is the characteristic magnitude of the matrix element $W(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)$ when all the four wave vectors $\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4$ are of the order K; $N = \int n_{\boldsymbol{k}} \mathrm{d}\boldsymbol{k}$ is the total wave action of the short-wave field. (Here the integration does not involve the neighbourhood of the point $\boldsymbol{p}_0$.) Thus,

$$\tau \simeq \omega^{-1} \theta^{-2} \tag{3.8}$$

where

$$\theta \simeq \frac{|W|N}{\omega} \simeq K^{5/2} N g^{-1/2} = K^2 \frac{\omega N}{g} \tag{3.9}$$

is the parameter of nonlinearity of the short-wave field; it is proportional to the mean-square slope of the short waves (cf. (3.6)).

In order for the diffusion equation to be valid, the diffusion time τ_0 should be much less than the characteristic time τ of close scale interactions. Otherwise, the cascade transfer of energy from the long-wave to the short-wave field would be more important than the diffusion of the spectrum $n_{\boldsymbol{k}}$. From (3.3) and (3.8) we have

$$\frac{\tau_0}{\tau} \simeq \frac{\omega A}{\mu B^2}\left(\frac{\theta}{\theta_0}\right)^2 = \frac{A(Kp_0)^{1/2}}{B^2}\left(\frac{\theta}{\theta_0}\right)^2. \tag{3.10}$$

We see from (3.10) that when the longitudinal spectral width A and the transversal spectral width B are of the same order ($A \simeq B$), the diffusion equation is valid when the long wave is much more nonlinear than the short waves, namely when

$$\frac{\theta}{\theta_0} \ll \left(\frac{(Kp_0)^{1/2}}{B}\right)^{1/2}.$$

The diffusion equation could also be valid when $A \ll B$ (see (3.10)).

One could also expect a significant contribution to the integral (2.1) from the region where one of the wave vectors $\boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4$ is 'close' to $\boldsymbol{p}_0$ while the other two are of order K. However, if p_0 is sufficiently smaller than K, the four-wave interactions degenerate into three-wave interactions, which are not allowed by the dispersion law (1.1).

3.3. *The modulation time T_0*

The diffusion mechanism would be important only if the time T during which the train of long waves passes a short wave is comparable to or bigger than the diffusion time τ_0. The time T cannot be less than the characteristic time of modulation of the wave train, which is defined by the frequency width Δ of this train:

$$T > T_0 \simeq \frac{1}{\Delta}.$$

This is similar to the uncertainty principle in quantum mechanics.

Using Taylor expansion, we find

$$\Delta \simeq \left|\frac{\partial \omega}{\partial p_x}(\boldsymbol{p}_0)\right| A + \left|\frac{\partial^2 \omega}{\partial p_y^2}(\boldsymbol{p}_0)\right| B^2 \simeq \mu_0 A + \frac{\mu_0}{2p_0} B^2 \tag{3.11}$$

(the first derivative $\partial\omega/\partial p_y$ vanishes at $\boldsymbol{p} = \boldsymbol{p}_0 = (p_0, 0)$ since the dispersion law ω depends only on the modulus of the vector $\boldsymbol{p}$; therefore, we need to take into account the second derivative $\partial^2\omega/\partial p_y^2$ when $B \gg A$.) From (3.3) and (3.11) we have

$$\frac{\tau_0}{T_0} \simeq \left(\frac{A^2}{B^2} + \frac{A}{2p_0}\right)\theta_0^{-2}. \tag{3.12}$$

In order for the diffusion mechanism to be important, the diffusion time τ should not be much bigger than the passage time T. This is possible if (i) $T \gg T_0$, i.e. the wave train is much longer than its modulation length, or (ii) $A \ll B$, i.e. the longitudinal spectral width is much smaller than the transversal spectral width (see (3.12), the parameter of nonlinearity is assumed small: $\theta_0 \ll 1$).

4. Conclusion

We see that the suggested mechanism of diffusion in the $\boldsymbol{k}$-plane could lead to the suppression of short waves by a train of long waves.

In order for the diffusion mechanism to be important, the interaction of the short waves with the long waves from the train should dominate the interaction of short waves among themselves. In the case of deep-water gravity waves, this requirement roughly means that the ratio (3.10) is much smaller than 1. This is possible if the long wave is much more nonlinear than the short waves: $\theta_0 \gg \theta$.

The diffusion will have a noticeable effect on the short-wave spectrum if the passage time T (during which the long-wave train passes a short wave) is not much smaller than the diffusion time τ_0. This requires that the wave train is much longer than its modulation length: $T \gg T_0$.

Also, the suggested mechanism can be effective (both of the above conditions can be satisfied: $\tau_0 \ll \tau$ and $\tau_0 < T$) if the train of long waves has a highly anisotropic distribution of wave action: its lonitudinal spectral width A (along $\boldsymbol{p}_0$) is much smaller than its transversal spectral width B (across $\boldsymbol{p}_0$), (see (3.10), (3.12)).

This diffusion mechanism is quite robust and can take place in various media. Actually, the short waves and the long waves can be of different physical nature and have different dispersion laws.

It is interesting that the long wave, in fact, supplies energy to short waves, but this gain of energy makes them dissipate faster. Indeed, on each of the curves (1.7), the total wave action (1.9) is conserved and is transported to the region of higher wavenumbers k, and therefore, higher energies $\omega(\boldsymbol{k})$ and larger dissipation.

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